Relationship Between Beta, Gamma and Bessel's Function

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1 Abstract

A relationship between Beta and Gamma Functions has been defined before.

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\Gamma(n) = (n-1)!$$

A relationship has now been defined for Beta, Gamma and Bessel Functions. Various properties of Beta, Gamma, and Bessel's Functions have been used to simplify the Bessel's summation down to a single term by building the relationship between Beta, Gamma and itself. Through the course of the paper, you will see how this relationship has been built and how it proves useful.

2 Definitions

Beta and Gamma Functions have a large purpose in Mathematics and Physics. These functions are special functions. Beta Function is a Euler Integral of the first kind. In mathematics, they are used to solve a large range of integrals. In physics, the beta function and the related gamma function is used to calculate and reproduce the scattering amplitude in terms of the Regge Trajectories.

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Gamma Function is one of the number of extensions of the factorial function with its argument shifted down by 1, to real and complex numbers. The gamma function was derived by Daniel Bernoulli.
The gamma function is a component in various probability distribution functions.
Bessel's Function are canonical solutions to the equation:

$$x^2y'' + xy' + (x^2 - n^2) = 0$$

Where c is is an arbitrary constant, and is defined as:

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{x}{2})^{2m+n}}{m!\ \Gamma(m+n+1)}$$

3 Derivation and Main Result

Initial Summation

$$J_{n+1}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\ x^{n+2r+1}}{2^{n+2r+1}\ r!\ \Gamma(n+r+2)}$$

$$J_{n+1}(x) = \frac{x^{n+1}}{2^{n+1}} \sum_{r=0}^{\infty} \frac{(-1)^r\ x^{2r}}{2^{2r}\ r!\ \Gamma(n+r+2)}$$

We know that,

$$\beta(n+1, r+1) = \frac{\Gamma(n+1)\Gamma(r+1)}{\Gamma(n+r+2)}$$

On substituting,

$$J_{n+1}(x) = \frac{x^{n+1}}{2^{n+1}} \sum_{r=0}^{\infty} \frac{(-1)^r \ x^{2r} \ \beta(n+1, r+1)}{2^{2r} \ r! \ \Gamma(n+1) \ \Gamma(r+1)}$$

$$J_{n+1}(x) = \frac{x^{n+1}}{2^{n+1}} \sum_{r=0}^{\infty} \frac{(-1)^r \ x^{2r} \ \beta(n+1, r+1)}{2^{2r} \ r! \ n! \ r!}$$

$$J_{n+1}(x) = \frac{x^{n+1}}{2^{n+1} \ n!} \sum_{r=0}^{\infty} \frac{(-1)^r \ x^{2r} \ \beta(n+1, r+1)}{2^{2r} \ r! \ r!}$$

We know that,

$$\beta(n+1, r+1) = \int_0^1 x^n (1-x)^r dx$$

By using integral by parts,

$$\beta(n+1, r+1) = x^n \int_0^1 (1-x)^r dx - \int_0^1 n x^{n-1} \int_0^1 (1-x)^r dx$$

If you plug this equation back, You get:

$$\beta(n+1, r+1) = \frac{x^n}{r+1} - n \int_0^1 \frac{x^{n-1} dx}{r+1}$$

On solving the integral,

$$\beta(n+1, r+1) = \frac{x^n - 1}{r+1}$$

Substituting it back into the summation,

$$J_{n+1}(x) = \frac{x^{n+1}\ (x^n - 1)}{2^{n+1}\ n!} \sum_{r=0}^{\infty} \frac{(-1)^r\ x^{2r}}{2^{2r}\ (r+1)\ r!\ r!}$$

$$J_{n+1}(x) = \frac{x^{n+1}\ (x^n - 1)}{2^{n+1}\ n!} \sum_{r=0}^{\infty} \frac{(-1)^r\ x^{2r}}{2^{2r}\ (r+1)!\ r!}$$

Multiplying and dividing by 2/x

$$J_{n+1}(x) = \frac{x^{n+1}\ (x^n - 1)}{2^{n+1}\ n!} \sum_{r=0}^{\infty} \frac{(-1)^r\ x^{2r+1}\ 2}{2^{2r+1}\ \Gamma(r+1+1)\ r!\ x}$$

$$J_{n+1}(x) = \frac{x^{n}\ (x^n - 1)}{2^{n}\ n!} \sum_{r=0}^{\infty} \frac{(-1)^r\ x^{2r+1}}{2^{2r+1}\ \Gamma(r+1+1)\ r!}$$

$$J_{n+1}(x) = \frac{x^{n}\ (x^n - 1)}{2^{n}\ n!} J_1(x)$$

Now when we plug n = 0, Equation won't be satisfied. Hence we replace back 1 with it's initial value.

$$J_{n+1}(x) = \frac{x^{n}\ (x^n - n \int_0^1 x^{n-1})}{2^{n}\ n!} J_1(x)$$

We know that,

$$\Gamma(n+1) = n! = \frac{\Gamma(n)}{\beta(n,1)}$$

We arrive at this final relationship,

$$J_{n+1}(x) = [\frac{x^n\ (x^n - n\int_0^1 x^{n-1})\ J_1(x)}{2^n}]\frac{\beta(n,1)}{\Gamma(n)}$$

where n $\in R - \{Z^-\}$

4 Bibliography

https://en.wikipedia.org/wiki/Beta_function
https://en.wikipedia.org/wiki/Gamma_function
https://en.wikipedia.org/wiki/Bessel_function